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Water Supply Planning Model for West Africa

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Introduction

In planning for a large region, it is important that the major elements of the system be adequately studied before more detailed analyses start. This is particularly the case in water resource development where there are often many alternative projects to consider, and many different detailed analyses that could be done. A methodology is needed that can screen the many alternatives and give guidance on what alternatives seem promising and need further analysis.

Fortunately, many regional water resource problems are amenable to mathematical modeling techniques that allow consideration of many alternatives and their interactions. Such a regional water resource planning model has been developed as part of a water supply planning framework for the arid, underdeveloped Sahel-Sudan region in West Africa. The six countries of the region are shown in Fig. 1. (The total area is approximately two-thirds that of the Continental United States.)

This paper presents a brief description of the region, a review of relevant literature, detailed formulation of the model, and an application illustrating its use. Even though many elements of the model are problem-specific, the overall modeling techniques are of value to engineers concerned with regional water resource issues in both developed and underdeveloped countries.

Sahel-Sudan Region

This section is a brief summary of the major features of the region. More information can be found in Major, et al. (5). The Sahel and Sudan regions are two vegetation and climatic belts south of the Sahara desert. The region

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is characterized by a short wet season (July–September) and a long dry season (October–June). Average annual rainfall varies from 12 in. (300 mm) to 35 in. (900 mm). Since the rainfall is in the form of squalls, it is very variable in both time and space. There are three major river basins in the region; each has headwaters in the tropical regions to the south. Like the precipitation, the discharges of these rivers show distinct seasonal effects and are highly variable.

Ground water is also available in the region. The principal types of aquifers are: (1) Deep fossil deposits; (2) ephemeral surface aquifers; (3) rock aquifers; and (4) recharged sedimentary formations. The region covers part of six countries. The population of the region is approx 23,000,000. The annual per capita income is $150. The majority of the population are either subsidence or cash crop farmers or nomadic herders. Only 2%–16% of the gross national product of each country is derived from the industrial sector.

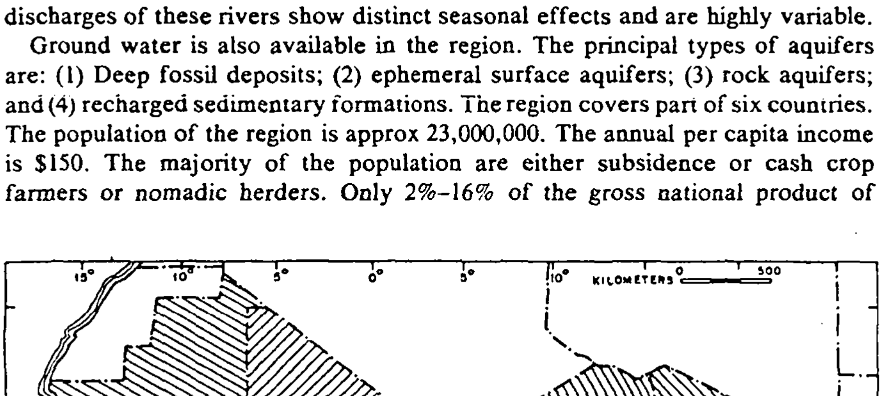

FIG. 1.—Study Region

The region recently suffered through a severe drought. The water resources planning model examined in this paper was developed as part of a methodology for long-term social and economic planning in the region.

LITERATURE REVIEW

Other regional water supply planning models have been developed. DeLucia and Rogers (1) developed a regional model to study water supply options to meet demand projections in the North Atlantic Region (NAR) of the United States. The region was divided into 50 planning subareas, and a variety of sources were modeled. Separable programming was used to determine the supply sources under critical period conditions. Heady, et al. (3) used a linear programming model to determine the least cost allocation of land and water resources

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included regulated surface water, desali
yield and average demands were used. In
the United States was divided into 223 ag
supply regions, and 27 market regions.

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study the optimum expansion of a regio
model selected the least cost set of so
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tion was used to link the outputs of the
maximized. Rausser and Willis (8) used
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the costs of recycling wastewater.

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in regional water supply planning, present
supply model. The MIP was used to m
options.

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MODEL FORMULATION

General Review.—The model is a MI
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The major constraints are that in each
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major decision variables in each subarea
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entire region, and some subareas have
it is necessary to include both large- a
the same model. It is not expected th
supplies and demands. The supply sou
runoff, water transfers, rivers, reservoir
and nonrecharged ground water. The wa

in the United States to meet future agricultural demands. The sources modeled included regulated surface water, desalination, and interbasin transfers. Safe yield and average demands were used. In one representative model application, the United States was divided into 223 agricultural-producing regions, 51 water supply regions, and 27 market regions.

Haimes and Nainis (2) formulated a set of interactive dynamic models to study the optimum expansion of a regional water supply system. The supply model selected the least cost set of sources that should be built each time period to meet demand. The demand model determined the value added to the regional economy given the quantity of water supplied. Multilevel decomposition was used to link the outputs of the models such that net benefits were maximized. Rausser and Willis (8) used a decomposition approach to study optimal expansion of a regional water supply system given the benefits and costs of research and development projects, multiple supply sources, and price-sensitive demands. Another model of interest is that of Narayanan, et al. (6), who used a nonlinear programming model to study water supply alternatives in urban areas. The emphasis was on wastewater treatment requirements and the costs of recycling wastewater.

Pugner, et al. (7), as part of an examination of the use of interactive models in regional water supply planning, presented a mixed integer programming (MIP) supply model. The MIP was used to model capital costs and build/no build options.

While elements of some of the aforementioned models are valuable in formulating a regional water supply planning model for the Sahel-Sudan region, none of the models are completely appropriate. Many of them are too complex and none of them consider the extreme wet and dry seasons in the Sahel-Sudan region, the seasonal demand variations, or the range of the supply sources in the region.

MODEL FORMULATION

General Review.—The model is a MIP optimization model. The application of the model to the region requires that each country be divided into three to six subareas and that the design year be divided into two seasons: the wet season (July through September), and the dry season (October through June). Possible objective functions are: (1) Minimization of the cost of supplying water to meet fixed demands; and (2) maximization of the net benefits of water use. The major constraints are that in each subarea and each season, the water demand must be met and that hydrologic continuity must be maintained. The major decision variables in each subarea are the operating policies of reservoirs, the capacities of the water supply sources, and, in the case of net benefits maximization, the magnitudes of the demands.

Since for planning purposes the model is intended to be applicable to the entire region, and some subareas have only small-scale development options, it is necessary to include both large- and small-scale development options in the same model. It is not expected that all subareas will have all types of supplies and demands. The supply sources include desalination, rainfall and runoff, water transfers, rivers, reservoirs, recycled water, and both recharged and nonrecharged ground water. The water demands include irrigation, municipalities, industries, rural use, and livestock.

The use of rainfall and runoff for water supply requires further explanation. In many parts of the Sahel-Sudan region, water is taken from small natural depressions, and runoff can be guided to particular sites by small earth or rock embankments. It is possible to increase the water available from these sources by compacting catchments, controlling vegetation, etc. The uses of this source in the region vary from the irrigation of small plots of land to principal water supplies for major cities. In the model, water taken from small natural depressions with no attempt to control or increase the runoff is referred to as "unimproved" use of runoff. Improved use of runoff refers to increasing the runoff available from an area by guiding it to particular sites or by ground changes, or both. It is also possible to store the precipitation and runoff for use in the following season. If evaporation suppressants are used, the storage is referred to in the model as "efficient" storage. If not, it is referred to as inefficient storage. In the model it is assumed that if a user makes the effort to improve his catchment, he will also make the effort to efficiently store the water. Therefore efficient storage is associated with improved catchments, and inefficient storage with unimproved catchments.

Two values of seasonal rainfall are modeled. One is the total seasonal rainfall in a subarea; this is available for storage. The other is seasonal sum of the monthly rainfall that is greater than 0.6 in. (15 mm). Runoff does not form in the region unless the monthly rainfall exceeds 0.6 in. (15 mm). Depending upon the aquifer, ground-water recharge is dependent upon one or both types of rainfall.

Constraints for Subareas.—Presented in this section is the detailed mathematical formulation of the model. Most of the variables have units of volume, with the exception of the following: capacities (volume/time), rainfall (depth), and areas (length^2). All the variables and coefficients are defined in Appendix II.

Supply Must Equal or Exceed Demand.—The total water demand in subarea i during season t must be met by the sources of supply: desalination DESAL_{it}; the remainder of the water transferred from subarea j to i during t, $\alpha_{\text{IM}_{jit}}\,\text{IM}_{jit}$ (a portion, $1-\alpha_{\text{IM}_{jit}}$, is lost to evaporation and seepage); surface water W_{it}; the precipitation or runoff that has been captured using unimproved means, ORSS_{it}; the runoff that has been captured using improved means, ORS_{it}; recycled water RECY_{it}; and the ground water pumped from aquifer k, GW_{ikt}. The demands include the demands for water use m in i during t, D_{imt}, the water exported from subarea i to l during t, IM_{ilt} and the artificial recharge to aquifer k during t, AR_{ikt}:

$$\text{DESAL}_{it} + \sum_{j} \alpha_{\text{IM}_{jit}}\,\text{IM}_{jit} + W_{it} + \text{ORS}_{it} + \text{ORSS}_{it} + \text{RECY}_{it}$$

$$+ \sum_{k} GW_{ikt} \geq \sum_{m} D_{imt} + \sum_{l} \text{IM}_{ilt} + \sum_{k} AR_{ikt} \quad \forall i, t \; \ldots\ldots\ldots\ldots \; (1)$$

Recycled Water.—Recycled water cannot exceed its sources (usually fractions β_{ik} of industrial and municipal water):

$$\text{RECY}_{it} \leq \sum_{k} \beta_{ik} D_{ikt} \quad \forall i, t \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; (2)$$

Supply Limitation.—A portion, δ_{ik}, of a
portions δ'_{ik} of a set of demands k'. This co
it is necessary to model some physical infeas
cultural infeasibility. The latter arises becau
traditional water supply sources that should no
Thus

$$\sum_{k} \delta_{ik}\,\text{Source}_{ikt} \leq \sum_{k} \delta'_{ik'}\,\text{Demand}_{ik't} \quad \forall i, t$$

Supply Requirement.—A portion, $\eta'_{ik'}$, of s
a portion, η_{ik}, of sources k:

$$\sum_{k} \eta_{ik}\,\text{Sources}_{ikt} \geq \sum_{k} \eta'_{ik'}\,\text{Demands}_{ik't} \quad \forall i$$

Distribution of Runoff.—Precipitation and
subareas $p\,(\Sigma_p \kappa_{pi}\text{RO}_{pit})$ contribute to: (1) Th
the subarea or to rivers and lakes in the sub
caught on unimproved and improved catchmen
Thus

$$\frac{1}{\epsilon_{1i}}\text{RUNOFF}_{it} + \frac{1}{\epsilon_{2i}}\text{SSAS}_{it} + \frac{1}{\epsilon_{3i}}\text{SSA}_{it} = \text{F}$$

$$+ \sum_{p} \kappa_{pi}\,\text{RO}_{pit} \quad \forall i, t \; \ldots\ldots\ldots\ldots$$

ϵ_{ji} = the efficiency of conversion of precipit
if an area had a runoff coefficient of 10%,
is the sum of the monthly rainfall in i durin
15 mm and thus causes runoff; and AREA 1 is th
occurs. It is usually the total area unless a lar
to irrigation. In this case, the area that is unde
the crops consume the rainfall. Included in
of precipitation lost to ground-water recharge. I
runoff formation (i.e., RUNOFF_{it} always equa
is changed to "less than or equal to."

Limitations on Use of Runoff.—Only so
subarea or entering the subarea from outside
The amount is limited because parts of the
low population densities. Thus

$$\frac{\text{SSSA}_{it}}{\epsilon_{2i}} + \frac{\text{SSA}_{it}}{\epsilon_{3i}} \leq \phi\,(\text{PREC}_{it}\,\text{AREA1}_{i} + \sum_{p}$$

Efficient Storage of Runoff.—The volume pr
$\text{SS}_{i,t}$, equals the remaining previously stored a
$1-\alpha$, is lost in evaporation) plus the inflow f
the outflow from storage, $\text{ORS}_{i,t}$. Thus

$$\alpha_{\text{SS}_{i,t-1}}\text{SS}_{i,t-1} + \text{SSA}_{it} = \text{SS}_{it} + \text{ORS}_{it} \quad \forall i, t$$

Supply Limitation.—A portion, δ_{ik}, of a set of sources k can only supply portions δ'_{ik} of a set of demands k'. This constraint and Eq. 4 are used when it is necessary to model some physical infeasibility, or more importantly, some cultural infeasibility. The latter arises because some of the population have traditional water supply sources that should not be changed or further developed. Thus

$$\sum_{k} \delta_{ik}\,\text{Source}_{ikt} \le \sum_{k} \delta'_{ik'}\,\text{Demand}_{ik't} \quad \forall i,t \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Supply Requirement.—A portion, $\eta'_{ik'}$, of some demands k' must be met by a portion, η_{ik}, of sources k:

$$\sum_{k} \eta_{ik}\,\text{Sources}_{ikt} \ge \sum_{k} \eta'_{ik'}\,\text{Demands}_{ik't} \quad \forall i,t \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Distribution of Runoff.—Precipitation and runoff entering subarea i from subareas $p\,(\Sigma_p \kappa_{pi}\text{RO}_{pit})$ contribute to: (1) The runoff that either flows out of the subarea or to rivers and lakes in the subarea, RUNOFF_{it}; and (2) water caught on unimproved and improved catchments for water supply SSAS_{it}, SSA_{it}. Thus

$$\frac{1}{\epsilon_{1i}}\text{RUNOFF}_{it} + \frac{1}{\epsilon_{2i}}\text{SSAS}_{it} + \frac{1}{\epsilon_{3i}}\text{SSA}_{it} = \text{PREC}_{it}\,\text{AREA1}_{i}$$

$$+ \sum_{p} \kappa_{pi}\,\text{RO}_{pit} \quad \forall i,t \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

ϵ_{ji} = the efficiency of conversion of precipitation to use j in i. For example, if an area had a runoff coefficient of 10%, ϵ_{1i} would be 0.10. Here PREC_{it} is the sum of the monthly rainfall in i during season t, which is greater than 15 mm and thus causes runoff; and AREA 1 is the area over which the precipitation occurs. It is usually the total area unless a large portion of a subarea is devoted to irrigation. In this case, the area that is under cultivation is subtracted because the crops consume the rainfall. Included in the AREA1_{i} term is the fraction of precipitation lost to ground-water recharge. In subareas where there is negligible runoff formation (i.e., RUNOFF_{it} always equals zero), the sign of the constraint is changed to "less than or equal to."

Limitations on Use of Runoff.—Only some of the runoff generated in the subarea or entering the subarea from outside can be utilized for water supply. The amount is limited because parts of the subarea are uninhabited, or have low population densities. Thus

$$\frac{\text{SSSA}_{it}}{\epsilon_{2i}} + \frac{\text{SSA}_{it}}{\epsilon_{3i}} \le \phi\,(\text{PREC}_{it}\,\text{AREA1}_{i} + \sum_{p} \kappa_{pi}\text{RO}_{pit}) \quad \forall i,t \quad \ldots\ldots (6)$$

Efficient Storage of Runoff.—The volume presently in efficient runoff storage, $\text{SS}_{i,t}$, equals the remaining previously stored amount, $\alpha_{\text{SS}_{i,t-1}}\text{SS}_{i,t-1}$ (a fraction, $1 - \alpha$, is lost in evaporation) plus the inflow from the catchment, SSA_{it}, minus the outflow from storage, $\text{ORS}_{i,t}$. Thus

$$\alpha_{\text{SS}_{i,t-1}}\text{SS}_{i,t-1} + \text{SSA}_{it} = \text{SS}_{it} + \text{ORS}_{it} \quad \forall i,t \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Inefficient Storage of Runoff.—Eq. 8 is the same as Eq. 7 except it applies to precipitation caught in unimproved catchments and stored inefficiently:

$$\alpha_{\mathrm{SSS}_{i,t-1}} \mathrm{SSS}_{i,t-1} + \mathrm{SSAS}_{it} = \mathrm{SSS}_{it} + \mathrm{ORSS}_{it} \quad \forall i, t \tag{8}$$

Flow of Runoff between Subareas.—The runoff entering subarea *i* that originates in subarea *j*, RO_{jit}, is a fraction, θ_{ji}, of the runoff in *j*:

$$\mathrm{RO}_{jit} = \theta_{ji} \mathrm{RUNOFF}_{jt} \quad \forall i, t \tag{9}$$

Continuity of Surface Water.—For a subarea with a river running through it, the surface water leaving the subarea to subarea *j*, I_{jit}, equals the inflows from upstream *k*, $\Sigma_k I_{kit}$, plus the runoff remaining after overland flow out of the subarea, $\Sigma_j (1 - \theta_{ij})$ RUNOFF_{it}, plus the tributary inflow, TRIB_{it}, plus the yield from upstream reservoirs *m*, $\Sigma_m Y_{imt}$, plus the return flow from uses *n* immediately upstream of *i*, $\Sigma_n \lambda_{i-1,n} D_{i-1,nt}$, minus (or plus) the seepage and evaporation losses (or baseflow), ER_{it}, minus the withdrawals from the river, W_{it}, minus the amount of recycled water coming from uses immediately upstream of *i* that would be returned to surface water if there is no recycling, μ_{i-1}, $\mathrm{RECY}_{i-1,t}$. Then

$$\sum_k I_{kit} + \sum_j (1 - \theta_{ij}) \mathrm{RUNOFF}_{it} + \mathrm{TRIB}_{it} + \sum_m Y_{imt} + \sum_n \lambda_{i-1,n} D_{i-1,nt}$$
$$- \mathrm{EF}_{it} - W_{it} - \mu_{i-1} \mathrm{RECY}_{i-1,t} = I_{jit} \quad \forall i, t \tag{10}$$

The term $\Sigma_m Y_{imt}$ is the sum of the possible reservoir yields using different operating policies during *t*. Due to 0 − 1 integer constraints (examined later), only one of these yields can be greater than zero. Possible operating policies are: (1) No reservoir at all (i.e., natural flows); (2) constant yield throughout the year; and (3) 20% of the yield in the wet season, 80% of the yield in the dry season.

Ground-Water Dynamics.—Aquifers of low storage capacity cannot provide carry-over storage of recharge. They receive recharge from a fraction of effluent flow, $\gamma_{ik} \mathrm{ER}_{it}$, or a fraction of the sum of monthly precipitation exceeding 15 mm, $o_{ik} \mathrm{PREC}_{it}$, or a fraction of the sum of the total precipitation, $\epsilon_{ik} \mathrm{TP}_{it}$, or all three.

$$\mathrm{GW}_{ikt} \leq \gamma_{ik} \mathrm{ER}_{it} + \epsilon_{ik} \mathrm{TP}_{it} + o_{ik} \mathrm{PREC}_{it} \quad \forall i, k, t \tag{11a}$$

Aquifers of large storage capacity can provide carry-over storage of recharge. The recharge sources include those previously mentioned as well as artificial recharge, the return recycled flow from users π_{imk} D_{it}, and a fraction of the runoff entering the subarea from subarea *j*, $\Sigma_j \rho_{jik} \mathrm{RO}_{jit}$. Thus

$$\sum_t \mathrm{GW}_{ikt} \leq A_{ik} \left(\sum_t \sigma_{ik} \mathrm{AR}_{ikt} + \sum_m \pi_{imk} D_{imk} + \sum_j \rho_{jik} \mathrm{RO}_{jit} \right.$$
$$\left. + \gamma'_{ik} \mathrm{EF}_{it} + \epsilon'_{ik} \mathrm{TP}_{it} + o'_{ik} \mathrm{PREC}_{it} - \mu'_{ik} \mathrm{RECY}_{it} \right) \quad \forall i, k \tag{11b}$$

The term $\mu'_{ik} \mathrm{RECY}_{it}$ has the same meaning as in Eq. 10. The term A_{ik} is a coefficient that reflects the ground-water withdrawal policy. If it is equal to

1.0, it means the annual volume taken from
recharge for the design conditions. If it is
decision-maker is willing to make greater
is the efficiency of the artificial recharge
recharge that actually reaches the aquifer).
recharge mechanisms operate for every typ
when an aquifer is not being recharged,
maximum amount it is possible to withdraw

Design Parameters.—The design values
preassigned:

$$\mathrm{PREC}_{it} = \overline{\mathrm{PREC}}_{it} \quad \forall i, t$$

$$\mathrm{TRIB}_{it} = \overline{\mathrm{TRIB}}_{it} \quad \forall i, t$$

$$\mathrm{TP}_{it} = \overline{\mathrm{TP}_{it}} \quad \forall i, t$$

$$\mathrm{EF}_{it} = \overline{\mathrm{EF}_{it}} \quad \forall i, t$$

All variables with a line over them are upper
or are design quantities.

Capacities of Sources.—The amount of wa
in any facility, or released from any rese
to the source or facility capacity. In many c
from volume/season to volume per hour o
factor, con_i. The conversion factor may als
factor to convert average demands to peak

con_i(yield from source *i* during *t*) ≤ (capacit

Limitation on Storage of Rainfall.—The a
of rainfall is limited:

$$\mathrm{SSM}_i + \mathrm{SSSM}_i \leq D_i \cdot \mathrm{AREA2}_i \quad \forall i$$

in which D_i = the maximum depth of the
the area in *i* that can be used for storage.

Capacities of Reservoirs.—There is a m
any reservoir depending upon its operating p

$$\mathrm{YM}_{ik} \leq \overline{\mathrm{YM}}_{ik} \cdot X_{ik} \quad \forall i, k$$

in which YM_{ik} = the annual reservoir yiel
YM_{ik} = a constant; and X_{ik} = a 0 − 1 int
policy selected by the model, $X_{ik} = 1$; it
next constraints, only one X_{ik} can be equa
YM_{ik} can be non-zero and positive.

Uniqueness of Reservoir Operating Polic
one operating policy:

$$\sum_k X_{ik} = 1 \quad \forall i$$

0, 1 Variables.—The value X_{ik} equals eithe

Non-negativity.—All other variables are gr

1.0, it means the annual volume taken from an aquifer cannot exceed the annual recharge for the design conditions. If it is greater than 1.0, it means that the decision-maker is willing to make greater use of the aquifer. The value σ_{ik} is the efficiency of the artificial recharge to the aquifer (i.e., the amount of recharge that actually reaches the aquifer). It should be noted that not all these recharge mechanisms operate for every type of aquifer in the region. In cases when an aquifer is not being recharged, an upper bound is assigned on the maximum amount it is possible to withdraw.

Design Parameters.—The design values of $PREC_{it}$, $TRIB_{it}$, and EF_{it} are preassigned:

$$PREC_{it} = \overline{PREC}_{it} \quad \forall i, t \qquad (12a)$$

$$TRIB_{it} = \overline{TRIB}_{it} \quad \forall i, t \qquad (12b)$$

$$TP_{it} = \overline{TP}_{it} \quad \forall i, t \qquad (12c)$$

$$EF_{it} = \overline{EF}_{it} \quad \forall i, t \qquad (12d)$$

All variables with a line over them are upper bounds of their respective variables or are design quantities.

Capacities of Sources.—The amount of water supplied from any source, stored in any facility, or released from any reservoir has to be less than or equal to the source or facility capacity. In many cases the units have to be converted from volume/season to volume per hour or volume per year by a conversion factor, con_i. The conversion factor may also contain a peak-to-average demand factor to convert average demands to peak demands:

$$con_i(\text{yield from source } i \text{ during } t) \le (\text{capacity of source } i) \quad \forall i, t \qquad (13)$$

Limitation on Storage of Rainfall.—The area that can be used for the storage of rainfall is limited:

$$SSM_i + SSSM_i \le D_i \cdot AREA2_i \quad \forall i \qquad (14)$$

in which D_i = the maximum depth of the storage facilities; and $AREA2_i$ = the area in i that can be used for storage.

Capacities of Reservoirs.—There is a maximum annual yield, $\overline{YM}_{ik}$ from any reservoir depending upon its operating policy k:

$$YM_{ik} \le \overline{YM}_{ik} \cdot X_{ik} \quad \forall i, k \qquad (15)$$

in which YM_{ik} = the annual reservoir yield needed as determined in Eq. 13; $\overline{YM}_{ik}$ = a constant; and X_{ik} = a 0 – 1 integer variable. If k is the operating policy selected by the model, X_{ik} = 1; it is zero otherwise. As set by the next constraints, only one X_{ik} can be equal to 1. Therefore only one of the YM_{ik} can be non-zero and positive.

Uniqueness of Reservoir Operating Policies.—The reservoir can only have one operating policy:

$$\sum_k X_{ik} = 1 \quad \forall i \qquad (16)$$

0, 1 Variables.—The value X_{ik} equals either 0 or 1.

Non-negativity.—All other variables are greater than or equal to 0.

Upper Bounds.—There can be bounds on the values of many variables based upon political, cultural, economic, and physical feasibility. Most of the bounds are upper bounds except for low flow constraints. Lower or fixed bounds on variables can also be set so that existing projects or uses are constrained to appear in the solution.

Objective Function.—The following objective function concerns maximization of net benefits. The terms OMR_{ik} and C_{ik} are the operation, maintenance, and replacement costs, and the capital cost of source k in subarea i. The term BNFT_{il} is the benefit associated with use l in sub-area i. Then

$$\begin{aligned}\text{Maximize } Z = &-\sum_i \Bigg[\sum_t \Bigg(\mathrm{OMR}_{i,\mathrm{ORS}_i}\,\mathrm{ORS}_{it} + \mathrm{OMR}_{i,\mathrm{ORSS}_i}\,\mathrm{ORSS}_{i,t}\\ &+ \mathrm{OMR}_{i,\mathrm{DESAL}_i}\,\mathrm{DESAL}_{it} + \mathrm{OMR}_{i,\mathrm{RECY}_i}\,\mathrm{RECY}_{it} + \mathrm{OMR}_{i,W_i}\,W_{it}\\ &+ \sum_j \mathrm{OMR}_{i,\mathrm{IM}_{ji}}\,\mathrm{IM}_{jit} + \sum_k \mathrm{OMR}_{i,\mathrm{AR}_{ik}}\,\mathrm{AR}_{ikt} + \sum_k \mathrm{OMR}_{i,\mathrm{GW}_{ik}}\,\mathrm{GW}_{ikt}\Bigg)\\ &+ \mathrm{OMR}_{i,\mathrm{SSM}_i}\,\mathrm{SSM}_i + \mathrm{OMR}_{i,\mathrm{SSSM}_i}\,\mathrm{SSSM}_i + \sum_e \mathrm{OMR}_{i,\mathrm{YM}_{ie}}\,\mathrm{YM}_{ie}\\ &+ C_{i,\mathrm{DESALM}_i}\,\mathrm{DESALM}_i + \sum_j C_{i,\mathrm{IMM}_{ji}}\,\mathrm{IMM}_{ji} + C_{i,\mathrm{WM}_i}\,\mathrm{WM}_i\\ &+ C_{i,\mathrm{RECYM}_i}\,\mathrm{RECYM}_i + \sum_k C_{i,\mathrm{GWM}_{ik}}\,\mathrm{GWM}_{ik} + \sum_k C_{i,\mathrm{ARM}_{ik}}\,\mathrm{ARM}_{ik}\\ &+ C_{i,\mathrm{SSM}_i}\,\mathrm{SSM}_i + C_{i,\mathrm{SSSM}_i}\,\mathrm{SSSM}_i + \sum_e C_{i,\mathrm{YM}_{ie}}\,\mathrm{YM}_{ie}\Bigg]\\ &+ \sum_i \sum_e \sum_t \mathrm{BNFT}_{ie}\,D_{iet} \qquad (17)\end{aligned}$$

All costs are in units of dollars per unit capacity or dollars per unit volume. The minimization of the cost of supply objective function is identical except that the benefit term, $\Sigma_i \Sigma_e \Sigma_t \mathrm{BNFT}_{ie}\, D_{iet}$, is omitted and the function is to be minimized.

Illustrative Application

The case study presented illustrates one of the many possible analyses that can be done with the model. The purpose of the case study was to determine the least-cost supply system to meet a set of projected demands. Other possible model runs are examined in Refs. 4 and 5.

It should be noted that in several cases data were assumed because available data were incomplete or inconsistent, or both, or because it was judged inappropriate to develop in detail such data given the illustrative goal of the case study.

The area chosen for study was southwestern Mauritania and northern Senegal, shown in Fig. 2. Its total area is 26,290 sq mile (68,100 km²). The area was

divided into three subareas based primarily
political boundaries [see Major, et al. (5)
of criteria for subarea selection].

Hydrology of Case Study Area.—The cli
most northern parts to Sudanese in the m
rainfall is 5.5 in. (139 mm) at Nouakchott and
runoff only forms in subarea M7, which l
other two subareas lie on permeable sedime
runoff forms.

The Senegal River flows through two sub
Mauritania and Senegal. At Bakel, the ave

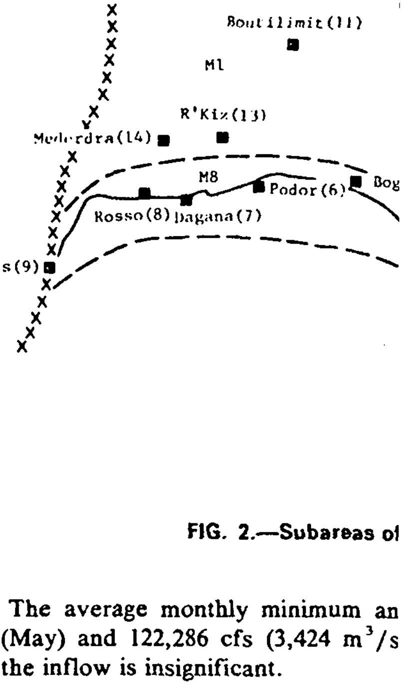

FIG. 2.—Subareas of

m³/s). The average monthly minimum an
m³/s) (May) and 122,286 cfs (3,424 m³/s
Bakel, the inflow is insignificant.

There are five major aquifers in the ar
and one rock. The sedimentary and alluv
M8 have water table depths of 10 ft (3 m
in subarea M7 has a water table depth o
M7 are also covered by small ephemeral
10 ft (3 m). Generally only the sedimentary
of ground water.

Water Demands.—The water demands
corresponding to a possible development sch
scheme assumed:

divided into three subareas based primarily upon hydrologic factors and natural political boundaries [see Major, et al. (5) for a more complete consideration of criteria for subarea selection].

Hydrology of Case Study Area.—The climate ranges from subdesert in the most northern parts to Sudanese in the most southern parts. Average annual rainfall is 5.5 in. (139 mm) at Nouakchott and 28 in. (712 mm) at Bakel. Significant runoff only forms in subarea M7, which lies on pre-Cambrian bedrock. The other two subareas lie on permeable sedimentary formations, and no significant runoff forms.

The Senegal River flows through two subareas and forms the border between Mauritania and Senegal. At Bakel, the average annual flow is 27,500 cfs (770 m^3/s). The average monthly minimum and maximum flows are 357 cfs (10 m^3/s) (May) and 122,286 cfs (3,424 m^3/s) (September), respectively. Below Bakel, the inflow is insignificant.

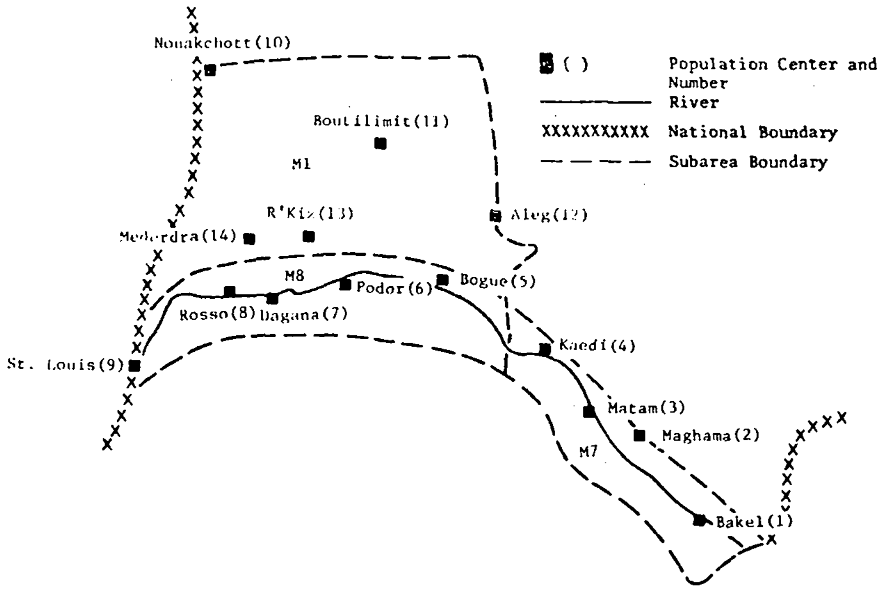

FIG. 2.—Subareas of Study Area

There are five major aquifers in the area; three sedimentary, one alluvial, and one rock. The sedimentary and alluvial formations in subareas M1 and M8 have water table depths of 10 ft (3 m)–200 ft (60 m). The rock aquifer in subarea M7 has a water table depth of 100 ft (30 m). Subareas M1 and M7 are also covered by small ephemeral aquifers with water table depths of 10 ft (3 m). Generally only the sedimentary formations provide overyear storage of ground water.

Water Demands.—The water demands used in the case study were those corresponding to a possible development scheme for the region. This development scheme assumed:

1. Economic and social development would result in large-scale irrigation in the valley and delta regions of the Senegal River [741,000 acres (300,000 ha) of irrigated land], livestock raising in southwestern Mauritania, and supporting industries located in the cities.

2. Livestock herds would remain in the north during the wet season to take advantage of the ephemeral rainfall and grasslands and then would be brought south in the dry season. An insignificant percentage of the northern rural population would make this migration with the herd so that the northern rural population would remain essentially constant.

3. Rural domestic (given Assumption 2), municipal, and industrial water use would be constant throughout the year. The irrigation demands would vary as shown in Table 1. No crops would be planted during the period November through February, but water would still be applied for salt-leaching purposes. The variation in livestock demand is also shown in Table 1.

TABLE 1.—Monthly Water Requirements for Irrigation and Livestock (Percentage of Total Annual Requirements)

Month (1)	Irrigation (all subareas) (2)	Livestock: Population centers 1-9, 13 (3)	Livestock: Population centers 10, 11, 12, 14 (4)
January	6.5	8.3	8.3
February	7.0	11.4	6.5
March	9.5	8.3	8.3
April	10.5	7.9	8.7
May	11.0	7.3	8.8
June	10.5	9.5	7.4
July	9.5	8.3	8.3
August	8.0	6.0	10.8
September	7.0	8.3	8.3
October	8.0	9.5	7.1
November	6.5	7.3	8.8
December	6.0	7.9	8.7

4. The low flow in the Senegal River in subarea M8 would be 3,570 cfs (100 m^3/s) so that salinity intrusion would be controlled in the delta and year-round navigation would be possible.

The demand quantities for the water uses are shown in Table 2.

Water Supplies.—Ground water, desalination, recycling, water transfers, rainfall and runoff, and water from the Senegal River were modeled as the possible sources to meet the projected demands of the foregoing development strategy. A reservoir representing the combined effects of several reservoirs was modeled upstream of Bakel.

Constraints were set requiring that during the dry season, the rock aquifer in subarea M7 and the alluvial aquifer in subarea M8 each supply 50% of the livestock and rural needs in their respective subareas. These constraints existed because of water use traditions.

The input values of rainfall and stream
return intervals. They were distributed t
average monthly values.

The reservoir had three possible operatin
out the year; (2) 20% of the yield in th
in the dry season (1/5–4/5 policy); and
constructed). A storage-yield curve corres
was developed for each of the first two ope
monthly streamflow data.

The operation, maintenance, and repla
supply were modeled. A planning horizo
discounted to the present with a rate of 10
subarea transfers, and reservoirs, all cos
was assumed because the construction, c
scattered projects such as wells built by
show significant economies of scale.

TABLE 2.—Annual Demand Qua

Subarea (1)	Irrigation* (2)	Livestoc (3)
M1	70,000	7,300
M7	878,680	2,550
M8	1,029,740	3,140

*This is the total demand. The net demand
in the actual model runs.

Note: 1 gal = 3.79 L.

For the nonlinear costs, cost curves
in the supply model to incorporate the n
of assuming the costs were linear but c
to the expected size of the structure. If,
was not as expected, the unit cost was adj

Case Study Results.—Table 3 presents
supply model, the seasonal volumes tak
three subareas, the capacities of the sou
costs.

As can be seen, the results are realistic:

1. Dividing the equivalent annual cost by
gives a unit water rate of approx $.03/1,0

2. Desalination is presently in use at
case study, desalination was not assume
has also been proposed to supply the ci
model has suggested (and more importantly
more detailed analyses have shown to be

The input values of rainfall and streamflow were drought values with 50-yr return intervals. They were distributed throughout the year based upon their average monthly values.

The reservoir had three possible operating policies: (1) Constant yield throughout the year; (2) 20% of the yield in the wet season and 80% of the yield in the dry season (1/5–4/5 policy); and (3) natural flows (i.e., no reservoir constructed). A storage-yield curve corresponding to a 0.02 probability of failure was developed for each of the first two operating policies using 50 yr of historical monthly streamflow data.

The operation, maintenance, and replacement (OMR) and capital costs of supply were modeled. A planning horizon of 50 yr was used. All costs were discounted to the present with a rate of 10%. Except for desalination, recycling, subarea transfers, and reservoirs, all costs were modeled as linear. Linearity was assumed because the construction, operation, and maintenance of small, scattered projects such as wells built by independent contractors would not show significant economies of scale.

TABLE 2.—Annual Demand Quantities, in millions of gallons

Subarea (1)	Irrigation[a] (2)	Livestock (3)	Municipal and industrial (4)	Rural (5)
M1	70,000	7,300	4,850	590
M7	878,680	2,550	230	760
M8	1,029,740	3,140	770	890

[a] This is the total demand. The net demand after the subtraction of rainfall was used in the actual model runs.

Note: 1 gal = 3.79 L.

For the nonlinear costs, cost curves were developed. The procedure used in the supply model to incorporate the nonlinearities was the iterative process of assuming the costs were linear but choosing the unit cost to correspond to the expected size of the structure. If, after solving the problem, the size was not as expected, the unit cost was adjusted and the problem resolved.

Case Study Results.—Table 3 presents the results of the application of the supply model, the seasonal volumes taken from each source in each of the three subareas, the capacities of the sources, and the total OMR and capital costs.

As can be seen, the results are realistic:

1. Dividing the equivalent annual cost by the annual quantity of water supplied gives a unit water rate of approx \$.03/1,000 gal (\$.007/m^3).

2. Desalination is presently in use at Nouakchott (in subarea M1). (In the case study, desalination was not assumed to exist at Nouakchott.) A pipeline has also been proposed to supply the city's future demands. Therefore, the model has suggested (and more importantly, not eliminated) sources that previous, more detailed analyses have shown to be favorable.

3. Relatively inexpensive sources such as rainfall, runoff, and surface water are used in large quantities where possible.

The sensitivity of the solution set to the hydrologic input conditions was studied. It was found that the solution set did not significantly change if average or severe drought hydrologic conditions were used. If other sensitivity analyses showed similar results, it would indicate that this configuration was a good starting point for more detailed analysis. Thus the model provides the types of answers that are of interest to engineers planning water resource development in this region.

TABLE 3.—Results of Case Study

Subarea (1)	Source (2)	Volume supplied, in millions of gallons		Required capacity, in gallons per minute (5)
		Wet season (3)	Dry season (4)	
M1	Desalination	1,450	4,080	10,890
	Catchments	1,130	0	Catchment area of 3,560 sq mile (8,200 km^2)
	Western Continental Terminal aquifer	0	2,630	13,210
	Eastern Continental Terminal aquifer	0	0	0
	Ephemeral aquifers	660	0	8,380
	Recycling	0	0	0
	Water transferred from M8 to R'Kiz	1,610	52,630	231,420
	Water transferred from M8 to Nouakchott and other Nodes	790	2,000	10,080
M7	Senegal River	153,160	648,160	2,850,080
	Catchments with storage	470	2,840	Catchment area of 280 sq mile (729 km^2)
	Rock aquifers	0	1,420	7,750
	Reservoir			7.6 × 10^6 acre-ft (9,320 × 10^6 m^3) Annual Yield, 1/5–4/5 Operating Policy
M8	Senegal River	229,470	826,320	3,633,460
	Catchments	580	0	Area of 1,040 sq mile (2,687 km^2)
	Alluvial aquifers	0	1,740	9,460

Note: Cost, in millions of dollars—OMR = 97.4; capital = 411.0; and total = 508.4, and 1 gal = 3.79 L.

SUMMARY AND CONCLUSIONS

A regional water supply planning model for the arid, underdeveloped Sahel-Sudan region has been presented. Its purpose is to provide a systematic framework

for the preliminary analysis of the region's w
uses mixed integer programming to determin
to meet fixed demands or the set of sourc
net benefits of water use. All types of proje

With rather a simple case study, it has bee
use is feasible and helpful to planners.

Further work on the theoretical develop
(1) The incorporation of multiple objectives ar
the effects of spatial and temporal aggrega
The latter was investigated by the writer (4) a
paper.

ACKNOWLEDGMENT

This research was performed at the Mas
(MIT) with the support of the United States A
(AID) and the MIT Department of Civil En
C. Major, David H. Marks, John C. Schaa
helpful comments.

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APPENDIX II.—NOTATION

The following symbols are used in this pa

A_{ik} = fraction of recharge to aq

for the preliminary analysis of the region's water supply alternatives. The model uses mixed integer programming to determine either the least-cost set of sources to meet fixed demands or the set of sources and demands that maximize the net benefits of water use. All types of projects can be modeled.

With rather a simple case study, it has been shown that the model's proposed use is feasible and helpful to planners.

Further work on the theoretical development of this model could include: (1) The incorporation of multiple objectives and stochastic effects; and (2) studying the effects of spatial and temporal aggregation upon the results of the model. The latter was investigated by the writer (4) and will be presented in a forthcoming paper.

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APPENDIX II.—NOTATION

The following symbols are used in this paper:

A_{ik} = fraction of recharge to aquifer k in i that can be withdrawn;

AR_{ikt} = amount of water artificially recharged to aquifer k in i during t;
$AREA1_i$ = area over which precipitation in i is assumed to act having accounted for ground-water recharge;
$AREA2_i$ = area in i that can be used for catchment runoff and storage facilities;
ARM_{ik} = capacity of artificial recharge facility in i to aquifer k;
C_{ij} = unit capital cost of source j in i;
con_i = conversion factor from seasonal use to capacity needed (includes peak-to-average demand increase factor);
D_i = maximum depth of storage facilities in i;
D_{ikt} = demand of type k in i during t;
$DESAL_{it}$ = amount of water supplied from desalination in i during t;
$DESALM_i$ = capacity of desalination plant in i;
EF_{it} = seepage and evaporation losses from surface water in i during t;
GW_{ikt} = amount of water pumped from aquifer k in i during t;
GWM_{ik} = capacity of ground-water pumping facility from aquifer k in i;
I_{jit} = surface water entering i from j during t (usually streamflow);
IM_{jit} = amount of water imported into i from j during t;
IMM_{ji} = capacity of water transfer facility from j to i;
OMR_{ij} = unit operation, maintenance, and replacement cost of source j in i;
ORS_{it} = amount of water taken from improved catchments and efficient storage of precipitation in i during t;
$ORSS_{it}$ = amount of water taken from unimproved catchments and inefficient storage of precipitation in i during t;
$PREC_{it}$ = sum of monthly precipitation in i that exceeds 15 mm/month during t;
$RECY_{it}$ = amount of water recycled in i during t;
$RECYM_i$ = capacity of recycling plant in i;
RO_{ij} = runoff from i to j during t;
$RUNOFF_{it}$ = runoff in i that enters surface water or flows out of the subarea during t, or both;
SS_{it} = amount of runoff or precipitation in efficient storage in i during t;
SSA_{it} = amount of precipitation caught in improved catchments in i during t;
$SSAM_i$ = capacity of improved catchments in i;
$SSAS_{it}$ = amount of precipitation caught in unimproved catchments;
SSM_i = capacity of efficient runoff and precipitation storage facility in i;
SSS_{it} = amount of runoff or precipitation in inefficient storage in i during t;
$SSSM_i$ = capacity of inefficient precipitation storage facilities in i;
TP_{it} = total amount of precipitation in i during t;
$TRIB_{it}$ = amount of tributary inflow into i during t;
W_{it} = amount of surface water withdrawn in i during t;

WM_i = capacity of surface water
X_{ik} = 0 – 1 integer variable
for reservoir in i;
Y_{ikt} = yield from reservoir in i us
YM_{ik} = annual yield from reservoi
α_i = fraction of water in source
seepage losses;
β_{ik} = amount of demand k that c
$\gamma_{ik}, \gamma'_{ik}$ = fraction of effluent flow E
$\delta_{ik}, \delta'_{ik'}$ = δ_{ik} of sources k can only s
ϵ_{1i} = efficiency on conversion of
in i;
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water in unimproved catch
ϵ_{3i} = efficiency of conversion o
water in improved catchme
$\eta_{ik}, \eta'_{ik'}$ = η_{ik} of sources k must sati
i;
θ_{ij} = fraction of $RUNOFF_{it}$ in i
κ_{pi} = fraction of runoff from sub
in i (some is lost to ground
λ_{in} = fraction of demand n in
i;
μ_i = fraction of recycled wate
water that normally returns
μ'_{ik} = fraction of recycled wate
water that normally return
ξ_{ik}, ξ'_{ik} = fraction of total precipitat
i;
o_{ik}, o'_{ik} = fraction of precipitation P
π_{imk} = fraction of demand m in
i;
ρ_{jik} = fraction of runoff from j t
σ_{ik} = efficiency of artificial rech
ϕ_i = fraction of total rainfall i
be captured in improved a

WM_i = capacity of surface water pumping facility in i;

X_{ik} = $0 - 1$ integer variable corresponding to operating policy k for reservoir in i;

Y_{ikt} = yield from reservoir in i using operating policy k during t;

YM_{ik} = annual yield from reservoir in i using operating policy k;

α_i = fraction of water in source i that is left after evaporation and seepage losses;

β_{ik} = amount of demand k that can be recycled;

$\gamma_{ik}, \gamma'_{ik}$ = fraction of effluent flow ER_{it} that recharges aquifer k in i;

$\delta_{ik}, \delta'_{ik'}$ = δ_{ik} of sources k can only supply $\delta'_{ik'}$ of demands k';

ϵ_{1i} = efficiency on conversion of precipitation PREC_{it} to RUNOFF_{it} in i;

ϵ_{2i} = efficiency of conversion of precipitation PREC_{it} to available water in unimproved catchment SSAS_{it} in i;

ϵ_{3i} = efficiency of conversion of precipitation PREC_{it} to available water in improved catchment SSA_{it} in i;

$\eta_{ik}, \eta'_{ik'}$ = η_{ik} of sources k must satisfy at least $\eta'_{ik'}$ of demands k' in i;

θ_{ij} = fraction of RUNOFF_{it} in i that enters subarea j;

κ_{pi} = fraction of runoff from subarea p to i that contributes to runoff in i (some is lost to groundwater recharge in i);

λ_{in} = fraction of demand n in i that returns to surface water in i;

μ_i = fraction of recycled water in i from nonconsumed demand water that normally returns to the surface water if not recycled;

μ'_{ik} = fraction of recycled water in i from nonconsumed demand water that normally returns to aquifer k in i if not recycled;

ξ_{ik}, ξ'_{ik} = fraction of total precipitation TP_{it} recharged to aquifer k in i;

o_{ik}, o'_{ik} = fraction of precipitation PREC_{it} recharged to aquifer k in i;

π_{imk} = fraction of demand m in i that is returned to aquifer k in i;

ρ_{jik} = fraction of runoff from j to i that recharges aquifer k in i;

σ_{ik} = efficiency of artificial recharge to aquifer k in i; and

ϕ_i = fraction of total rainfall in i and runoff entering i that can be captured in improved and unimproved catchments.